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Gravitational Theory Based on Relativistic Mechanics

Cui Douxing[1]

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Introducing a metric space, we propose a gravitational theory in which the form of the basic equations of mechanics, the field equations, and the equations of motion are the same as that of the corresponding equations in electrodynamics. The theory reveals a very close relation between the gravitational and electromagnetic fields. Finally, we consider the field due to an arbitrarily moving mass point.

1. INTRODUCTION

In relativistic mechanics, the equations of motion for a mass point have the form

$$\frac{d}{dt}(mu^i) = \beta^{-1}G^i \tag{1}$$

where G^i is a four-vector known as the Minkowski force, $u^i = dx^i/d\tau$, $d\tau = \beta^{-1}dt$, $\beta = (1 - v^2/c^2)^{-1/2}$, $x^0 = ict$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$.

On the other hand, u^i can be considered as functions of time and coordinates x^i, $i = 1, 2, 3$, and we have

$$\begin{aligned}\frac{d}{dt}(mu^i) &= m\left(\partial_t u^i + \sum_{j=1}^{3} \dot{x}^j\partial_j u^i\right)\\ &= m\left[\partial_t u^i + \sum_{j=1}^{3} (\dot{x}^j\partial_j u^i - \dot{x}^j\partial_i u^j) + \sum_{j=1}^{3} \dot{x}^j\partial_i u^j\right]\\ &= m\left[c^2\partial_i\beta + \partial_t u^i + \sum_{j=1}^{3} \dot{x}^j(\partial_j u^i - \partial_i u^j)\right], \qquad i = 0, 1, 2, 3 \end{aligned} \tag{2}$$

[1] Changchun Artificial Satellite Observatory, Chinese Academy of Sciences, P.O. Box 1067, Changchun, Kirin, China.

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since

$$\sum_{j=1}^{3} \dot{x}^j \partial_i u^j = \sum_{j=1}^{3} \beta^{-1} u^j \partial_i u^j = \frac{1}{2} \beta^{-1} \partial_i (\beta^2 v^2) = c^2 \partial_i \beta$$

Here we use ∂_t, ∂_j, and $\dot{x}^j$ for $\partial/\partial t$, $\partial/\partial x^j$, and dx^j/dt, respectively. From (2) we have

$$\frac{d}{dt}(c^2\beta) = \mathbf{V} \cdot \mathbf{L}, \qquad \frac{d}{dt}(m\mathbf{u}) = m(\mathbf{L} + \mathbf{V} \times \mathbf{M}) \tag{3}$$

where

$$\mathbf{L} = -\text{grad}\, \Phi - \partial_t A, \qquad \Phi = -c^2\beta, \qquad A_b = -u^b, \qquad b = 1, 2, 3$$

$$M = \text{rot}\, \mathbf{A}, \qquad \mathbf{A} = (A_1, A_2, A_3), \qquad \mathbf{V} = (\dot{x}^1, \dot{x}^2, \dot{x}^3), \qquad \mathbf{u} = (u^1, u^2, u^3) \tag{4}$$

The unknown functions **L** and **M** can be obtained from a system of partial differential equations, namely the field equations, with some additional conditions.

As shown above, equations (1) can be written in the form of equations (3), and the relativistic force is expressed as

$$\beta^{-1} G^0 = \frac{im}{c} \mathbf{V} \cdot \mathbf{L}, \qquad \beta^{-1}(G^1, G^2, G^3) = m(\mathbf{L} + \mathbf{V} \times \mathbf{M})$$

Inasmuch as the form of the right-hand side of equations (3) is the same as that of the Lorentz force in electrodynamics and the definitions of **L** and **M** in (4) yield the half of the Maxwell field equations

$$\text{rot}\, \mathbf{L} + \partial_t \mathbf{M} = 0, \qquad \text{div}\, \mathbf{M} = 0 \tag{5}$$

it is natural to believe **L** and **M** also satisfy the other half of the Maxwell field equations (for empty space)

$$\text{div}\, \mathbf{N} = 0, \qquad \text{rot}\, \mathbf{K} - \partial_t \mathbf{N} = 0 \tag{6}$$

where $\mathbf{N} = \epsilon_0 \mathbf{L}$, $\mathbf{K} = (1/\mu_0)\mathbf{M}$, $\epsilon_0\mu_0 = 1/c^2$, and ϵ_0 and μ_0 are parameters.

Sections 2 and 3 are devoted to the derivation of equations (6).

2. THE GEOMETRY

It is well known that universal gravitation does not fit into the framework of uniform Galilean space. It is possible, however, to base a theory of universal gravitation on the idea of abandoning the uniformity of space as a whole and attributing to space only a certain kind of uniformity on an infinitesimal scale. There still exists on the infinitely small scale a uniformity like the one expressed by the Lorentz transformations. That this is so is connected with the fact that in the vicinity of a given point of space a gravitational field can be imitated by a field of acceleration (the principle of equivalence). It is obvious that at least one nonuniform space exists, that is, real space, and in which the form of the expression for the space-time interval between two infinitesimally near world points is preserved under Lorentz transformations. In the general case of Riemannian geometry there are no transformations of the coordinates that leave invariant the coefficients $g_{\alpha\beta}(x^0, x^1, x^2, x^3)$ of the quadratic form for the squared infinitesimal distance invariant. However, since in the vicinity of a given point of space the gravitational field can be imitated by a field of acceleration, it is natural to write $g_{\alpha\beta}$ in the form

$$g_{\alpha\beta} = g_{\alpha\beta}(u^0, u^1, u^2, u^3); \qquad u^i = \frac{dx^i}{d\tau} = u^i(x^0, x^1, x^2, x^3), \quad \alpha, \beta, i = 0, 1, 2, 3$$

where $d\tau$ is a scalar, and $g(u)$ is an invariant form. The invariance of the expression for $g_{\alpha\beta}$ under Lorentz transformations is expressed by $g'_{\alpha\beta} = g_{\alpha\beta}(u'^0, u'^1, u'^2, u'^3)$, $u'^i = dx'^i/d\tau$. This is a necessary condition of the principle of relativity.

The special theory of relativity formally characterizes relativistic physics by the invariance of the expression for the space-time interval between two world points

$$S_{12}^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

The invariance of this quadratic form of the coordinate differences restricts the group of all conceivable linear transformations of the four coordinates x, y, z, and t to that of the Lorentz transformations. We shall show that it is possible to introduce a nonuniform space such that the expression for the space-time interval between two infinitesimally near world points is Lorentz invariant under the above-mentioned meaning. If we put $x^0 = ict$, $i = \sqrt{-1}$, and $x = x^1$, $y = x^2$, $z = x^3$, then the inverse Lorentz transformation is of the form

$$x^i = c^i_k x'^k \quad \text{or} \quad dx^i = c^i_k dx'^k, \qquad c^h_i c^h_k = \delta_{ik},$$
$$\delta_{ik} = \begin{cases} 0, & i \neq k = 0, 1, 2, 3 \\ 1, & i = k \end{cases} \tag{7}$$

With the aid of (7) we obtain

$$\beta^{-2}(dx^0)^2 = \beta'^{-2}(dx'^0)^2 \tag{8}$$

where

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$\beta' = \left(1 - \frac{v'^2}{c^2}\right)^{-1/2}$$

$$v^2 = \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2$$

$$v'^2 = \left(\frac{dx'^1}{dt'}\right)^2 + \left(\frac{dx'^2}{dt'}\right)^2 + \left(\frac{dx'^3}{dt'}\right)^2$$

It follows from (8) that $\beta^{-1}dt = \beta'^{-1}dt'$. If we put $d\tau = \beta^{-1}dt = \beta'^{-1}dt'$, $u^i = dx^i/d\tau$, and $u'^i = dx'^i/d\tau$, then from (7) we have

$$u^i = c^i_k u'^k \tag{9}$$

We write the fundamental quadratic form as

$$ds^2 = g_{ij}dx^i dx^j \tag{10}$$

The coefficients g_{ij} must be so chosen that the form of (10) is preserved under Lorentz transformations. If we impose this condition on the coefficients g_{ij}, then

$$g_{ij} = c_1\delta_{ij} + c_2 u^i u^j \tag{11}$$

where c_1 and c_2 are constant. The substitution from (7) converts (10) into $ds^2 = g_{ij}c^i_k c^j_h dx'^k dx'^h$. Thus,

$$g'_{kh} = g_{ij}c^i_k c^j_h = (c_1\delta_{ij} + c_2 u^i u^j)c^i_k c^j_h$$
$$= c_1 c^i_k c^i_h + c_2 c^i_m c^j_n c^i_k c^j_h u'^m u'^n = c_1\delta_{kh} + c_2 u'^k u'^h$$

Clearly, the form of (11) is preserved and therefore the form of (10) is preserved under Lorentz transformation if g_{ij} is defined by (11). On the other hand, if g_{ij} is defined by (11), then it can be proved that the invariance of the form of (10) restricts the group of all conceivable linear transformations to that of the Lorentz transformations. The proof is quite elementary and will be omitted here. Since $(u^i)^2 = -c^2$, the contravariant metric tensor can be written as follows:

$$g^{ij} = \frac{1}{c_1}\delta^{ij} + \frac{c_2}{c_1(-c_1 + c_2c^2)}u^i u^j \tag{12}$$

Inasmuch as $ds^2 = g_{ij}dx^i dx^j = (-c_1 + c_2c^2)c^2 d\tau^2$ and $g = |g_{ij}| = c_1^3(c_1 - c_2c^2)$, where g denotes the determinant of the covariant metric tensor, it is natural to restrict the choice of constants c_1 and c_2 by conditions which ensure that $ds^2 = c^2 d\tau^2$ and $g = -1$. Thus, $c_1 = 1$, $c_2 = 2/c^2$, and

$$g_{ij} = \delta_{ij} + \frac{2}{c^2} u_i u_j, \qquad g^{ij} = \delta^{ij} + \frac{2}{c^2} u^i u^j \tag{13}$$

From (9) we know that u^i is a contravariant vector, namely, four-dimensional velocity. The covariant vector is written as $u_i = g_{ij} u^j = -u^i$. In order to formulate field equations and the equations of motion in this geometry, we introduce some scalars and vectors.

(1) If $u_{\beta,\alpha}$ denotes the covariant derivative of the vector u_β, then $F_{\alpha\beta} = u_{\beta,\alpha} - u_{\alpha,\beta} = \partial_\alpha u_\beta - \partial_\beta u_\alpha$ is a tensor and $F_\alpha = u^\beta F_{\beta\alpha}$ is a vector. Further,

$$F_\alpha F^\alpha = u^\beta F_{\beta\alpha} u^\mu F_{\mu\gamma} g^{\gamma\alpha} = u^\alpha u^\beta \partial_\alpha u_\mu \partial_\beta u_\mu \tag{14}$$

is a scalar.

(2) If we denote the Christoffel symbol, the second-rank curvature tensor, and the curvature scalar by Γ^h_{ik}, R_{ik}, and R, respectively, then since $\sqrt{-g} = 1$, we have

$$\begin{aligned}\sqrt{-g}R = g^{ik} \cdot R_{ik} &= g^{ik}(\partial_l \Gamma^l_{ik} - \partial_k \Gamma^l_{il} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}) \\ &= \sqrt{-g}G + \frac{4}{c^2} \partial_\alpha (u^\beta \partial_\beta u_\alpha)\end{aligned} \tag{15}$$

where

$$\begin{aligned}\sqrt{-g}G = \frac{4}{c^2} \partial_\alpha u_\beta \cdot \partial_\beta u_\alpha &- \frac{2}{c^2} \partial_\alpha u_\alpha \cdot \partial_\beta u_\beta - \frac{2}{c^2} \partial_\alpha u_\beta \cdot \partial_\alpha u_\beta \\ &- \frac{2}{c^4} u^\alpha u^\beta \partial_\alpha u_\gamma \cdot \partial_\beta u_\gamma\end{aligned} \tag{16}$$

3. THE FIELD EQUATIONS AND THE EQUATIONS OF MOTION

Since

$$\begin{aligned} u^\beta \partial_\alpha u_\beta &= -u^\beta \partial_\alpha u^\beta = -\tfrac{1}{2}\partial_\alpha (u^\beta)^2 = \tfrac{1}{2}\partial_\alpha c^2 = 0 \\ \frac{du^\alpha}{d\tau} &= u^\beta \partial_\beta u^\alpha = -u^\beta \partial_\beta u_\alpha \end{aligned}$$

it follows that

$$\frac{d^2 x^\alpha}{d\tau^2} = u^\beta F_{\alpha\beta} \tag{17}$$

Equation (17) denotes the equations of motion and yields equations (3).

The field equations for empty space can be represented as the Euler–Lagrange equations of a variational problem. The integral is a four-dimensional integral

$$I = \int \Lambda \sqrt{-g}\, d\Omega, \qquad \delta I = 0, \qquad d\Omega = dx^0 dx^1 dx^2 dx^3 \tag{18}$$

The geometrical properties are indissolubly linked with the field of acceleration. As was shown above, R and $F_\alpha F^\alpha$ are scalars concerned with the geometrical properties and the field of acceleration, respectively. Thus it is natural to define Λ as $\Lambda = R + kF_\alpha F^\alpha$. The constant k will be determined later. With the aid of (15), the variation of the integral (18) can be represented as follows:

$$\begin{aligned}\delta I &= \delta \int (R + kF_\alpha F^\alpha)\sqrt{-g}\, d\Omega \\ &= \delta \int (G + kF_\alpha F^\alpha)\sqrt{-g}\, d\Omega + \frac{4}{c^2}\int \partial_\alpha[\delta(u^\beta \partial_\beta u_\alpha)]\, d\Omega\end{aligned}$$

The last integral vanishes.

We have

$$\delta I = \int \left[\frac{\partial (G + kF_\alpha F^\alpha)\sqrt{-g}}{\partial u_i} - \partial_l \frac{\partial (G + kF_\alpha F^\alpha)\sqrt{-g}}{\partial(\partial_l u_i)} \right] \delta u_i\, d\Omega = 0$$

and therefore

$$\frac{\partial (G + kF_\alpha F^\alpha)\sqrt{-g}}{\partial u_i} - \partial_l \frac{\partial (G + kF_\alpha F^\alpha)\sqrt{-g}}{\partial(\partial_l u_i)} = 0 \tag{19}$$

Substitution from (14) and (16) converts (19) into

$$\begin{aligned}\left(\frac{4}{c^4} - 2k\right)(u^\alpha u^\beta \partial_\alpha \partial_\beta u_i + u^\alpha \partial_\alpha u_\beta \partial_i u_\beta - u^\alpha \partial_\alpha u_i \partial_\beta u_\beta - u^\alpha \partial_\alpha u_\beta \partial_\beta u_i) \\ + \frac{4}{c^2}\partial_\alpha F_{\alpha i} = 0\end{aligned}$$

Taking $k = 2/c^4$, we obtain the field equations for empty space

$$\partial_\alpha F_{\alpha i} = 0 \tag{20}$$

If we introduce two parameters ϵ_0 and μ_0, $\epsilon_0\mu_0 = 1/c^2$, then equation (20) yield equations (6).

The derivation of the field equations in the interior of matter will be omitted here.

4. FIELD DUE TO A MOVING MASS POINT

As was shown above, the gravitational field equations are expressed in the form of Maxwell's field equations in electrodynamics. Thus, in the Lienard–Wiechert potential (Pathria, 1963) let the origin of the system S with respect to which the charge has velocity u coincide with the point of observation; we obtain Φ and $\mathbf{A}$ of a mass point moving arbitrarily:

$$\Phi = -\frac{\mu}{s}, \qquad \mathbf{A} = \frac{\mu V}{c^2 s} \tag{21}$$

where $s = r + \mathbf{r}\cdot\mathbf{V}/c$, $-\mathbf{V} = -d\mathbf{r}/dt'$ is the velocity of the mass point with respect to the system S fixed at the point of observation, $t' = t - r(t')/c$, $\mu = km$, k is the Newtonian constant of gravitation, and $\mathbf{r}$ is the radius vector directed from the position of the mass point at time t' to the point of observation. Moreover, $\mathbf{r}$, r, and $\mathbf{V}$ must be taken as corresponding to the time t' and not to the time t. From these, one can directly calculate the field intensities $\mathbf{L}$ and $\mathbf{M}$ by means of the relations (4):

$$\mathbf{L} = -\frac{\mu r}{c^2 s^2}\dot{\mathbf{V}} - \frac{\mu}{s^3}\left(\mathbf{r} + \frac{r}{c}\mathbf{V}\right)\left(1 - \frac{v^2}{c^2} - \frac{\mathbf{r}\cdot\dot{\mathbf{V}}}{c^2}\right) \tag{22}$$

$$\mathbf{M} = -\frac{\mu}{c^3 s^2}(\mathbf{r}\times\dot{\mathbf{V}}) - \frac{\mu}{c^2 s^3}(\mathbf{r}\times\mathbf{V})\left(1 - \frac{v^2}{c^2} - \frac{\mathbf{r}\cdot\dot{\mathbf{V}}}{c^2}\right)$$

where $\dot{\mathbf{V}} = d\mathbf{V}/dt'$. On the other hand, by taking the Lorentz gauge we can obtain (21) directly from the field equations (6). Expanding $\mathbf{r}$, r, and $\mathbf{V}$ into Taylor series based on the point $t' = t$, we have

$$(s(t'))^{-1} = \left[\frac{1}{r} - \frac{\mathbf{r}\cdot\mathbf{V}}{cr^2} + \frac{(\mathbf{r}\cdot\mathbf{V})^2}{c^2 r^3}\right]_{t'} + O(c^{-3})$$

$$= \left[\frac{1}{r} + \frac{v^2}{2c^2 r} + \frac{\mathbf{r}\cdot\dot{\mathbf{V}}}{2c^2\mathbf{r}} - \frac{(\mathbf{r}\cdot\mathbf{V})^2}{2c^2 r^3}\right]_t + O(c^{-3})$$

$$\mathbf{V}(t') = \left(\mathbf{V} - \frac{r}{c}\dot{\mathbf{V}}\right)_t + O(c^{-2})$$

and it follows that

$$\mathbf{L} = \left[\left(-\frac{\mu}{r^3} - \frac{\mu v^2}{2c^2 r^3} - \frac{\mu(\mathbf{r}\cdot\dot{\mathbf{V}})}{2c^2 r^3} + \frac{3\mu(\mathbf{r}\cdot\mathbf{V})^2}{2c^2 r^5}\right)\mathbf{r} - \frac{\mu}{2c^2 r}\dot{\mathbf{V}}\right]_t + O(c^{-3})$$

$$\mathbf{M} = \left[-\frac{\mu}{c^2 r^3}(\mathbf{r}\times\mathbf{V})\right]_t + O(c^{-3}) \tag{23}$$

It is simple to see that the interval of the mass points proper time $d\bar{t}$ is $dt/\sqrt{1 - v^2/c^2}$, where

$$v^2 = \left(-\frac{d\mathbf{r}}{dt}\right)\cdot\left(-\frac{d\mathbf{r}}{dt}\right) = \left(\frac{d\mathbf{r}}{dt}\right)\cdot\left(\frac{d\mathbf{r}}{dt}\right)$$

and from (3) we have

$$\left(\frac{du}{d\bar{t}}\right) = \mathbf{L} + \mathbf{V} \times \mathbf{M} \tag{24}$$

We checked that **L** and **M** expressed in (23) satisfy equations (5) and (6).

Substitution from (23) into (24) yields

$$\frac{d^2}{dt^2}\mathbf{r} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{F}$$

where

$$\mathbf{F} = \left(-\frac{3\mu(d\mathbf{r}/dt)^2}{2c^2r^3} + \frac{\mu^2}{c^2r^4} + \frac{3\mu(\mathbf{r}\cdot d\mathbf{r}/dt)^2}{2c^2r^5}\right)\mathbf{r} + \frac{2\mu(\mathbf{r}\cdot d\mathbf{r}/dt)}{c^2r^3}\frac{d\mathbf{r}}{dt} \tag{25}$$

We neglect the terms of $O(c^{-3})$ in (25).

If R, S, and W denote the orthogonal components of the perturbing acceleration **F** (R radially outward, S in the orbital plane orthogonal to R and directed with the motion, and W orthogonal to the orbit in such a sense that RSW forms a right-handed triad), then we have

$$R = -\frac{3\mu v^2}{2c^2r^2} + \frac{\mu^2}{c^2r^3} + \frac{7\mu(\mathbf{r}\cdot\dot{\mathbf{r}})^2}{2c^2r^4}$$

$$S = \frac{2\mu(\mathbf{r}\cdot\dot{\mathbf{r}})v^2}{hc^2r^2} - \frac{2\mu(\mathbf{r}\cdot\dot{\mathbf{r}})^3}{hc^2r^4}$$

$$W = 0$$

where $h = (\mu p)^{1/2}$, $p = a(1 - e^2)$, a is the semiaxis, and e is the eccentricity.

By using a convenient method (Cui, 1984) based on the planetary equations (Groves, 1965) we obtain the displacement of the perihelion after one period of revolution of the planet:

$$\Delta\omega = \frac{6\mu\pi}{c^2p}$$

This result is the same as the corresponding result of the general theory of relativity.

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